FT/UCM–7–96

On the "gauge" dependence of the toplogical sigma model beta functions

Luis Alvarez-Cónsul†

Departamento de Física Teórica, Universidad Autónoma de Madrid, Cantoblanco,28049 Madrid, Spain

C. P. Martín*

Departamento de Física Teórica I, Universidad Complutense, 28040 Madrid, Spain

We compute the dependence on the classical action "gauge" parameters of the beta functions of the standard topological sigma model in flat space. We thus show that their value is a "gauge" artifact indeed. We also show that previously computed values of these beta functions can be continuously connected to one another by smoothly varying those "gauge" parameters.

PCAS numbers 11.10.Gh, 11.10.Kk, 11.10.Lm

The topological sigma model introduced in ref. [1] is a particular instance of Topological Field Theory (see [2] for a review), and it lacks, therefore, physical propagating local degrees of freedom. As a result, the observables of the model are expected to be ultraviolet finite. And yet, the beta functions of the model for the classical action of ref. [1] turns out not to vanish [3]. The solution to this riddle has been advanced by the authors of ref. [4]. These authors suggest that a non-vanishing beta function is merely a "gauge" artifact, which has therefore no bearing on the value of the observables of the model. They back their argument by introducing a classical action for the topological sigma model that continuously connects, by means of a "gauge" parameter, say, κ_1, the action of ref. [1], which demands $\kappa_1 = 1$, with the "delta gauge" action, which corresponds to $\kappa_1 = 0$. They then go on and compute the one-loop contributions to the effective action for the "delta gauge" classical action. These contributions are ultraviolet finite so that the one-loop beta functions for the delta-gauge action vanish. The issue, however, has not been settled yet since it has not been shown that the non-vanishing beta functions obtained in ref. [3] can be continuously make to vanish by sending κ_1 to zero. It may well happen that the theories obtained for $\kappa_1 = 1$ and $\kappa_1 = 0$ are not the same quantum theory in spite of the fact that their classical actions differ by a BRSTlike-exact term: anomalies may turn up upon quantization. It is thus necessary to compute the beta functions for arbitrary values of κ_1 and show that these functions connect continuously the non-vanishing beta functions obtained in ref. [3] with the vanishing beta functions of ref. [4]. The purpose of this paper is to carry out this computation and show that the beta functions depend on κ_1 as expected.

† email: Lalvarez@delta.ft.uam.es

* email: carmelo@eucmos.sim.ucm.es

We would like to do our computations by using the superfield formalism introduced in ref. [3]. The first issue to tackle will thus be the existence of a superfield action that matches the action in ref. [4]. The latter action is obtained by setting $\kappa_2 = 1$ in the following equations

$$\mathcal{S}(\kappa_1,\kappa_2) \,=\, -i\{Q, \mathrm{V}(\kappa_1,\kappa_2)\}\;, \tag{1}$$

$$\mathrm{V}(\kappa_1,\kappa_2) \,=\, \int d^2\sigma\, \rho^{\alpha i}\Big(-\frac{\kappa_1}{4}H_\alpha^{\;i} + \kappa_2\, \partial_\alpha u^i\Big)\, g_{ij}\;. \tag{2}$$

Let us display the field content of the model. First, we introduce the fields $u^i(\sigma)$, which have conformal spin zero. $u^i(\sigma)$ describing (locally) a map f from a Riemann surface, Σ, to an almost complex riemannian manifold M; the almost complex extructure on M being denoted by $J^i_{\;j}$. Notice that the symbol g_{ij} stands for the hermitian metric on M with regard to $J^i_{\;j}$; σ denotes a point in Σ. Secondly, we define the anticonmuting field $\chi^i(\sigma)$ of conformal spin cero to be geometrically interpreted as a section of the pullback by f of the tangent bundle to M; this pullback being denoted by $f^*(T)$. We need two more fields. Let us call them $\rho^{\alpha i}$ and $H^{\alpha i}$, respectively. They both have conformal spin one, and, they both give rise to sections of the bundle of one forms over Σ with values on $f^*(T)$. The fields $\rho^{\alpha i}$ and $H^{\alpha i}$ are anticommuting and commuting objects, respectively, and they obey the selfduality constraints

$$\rho^{\alpha i} \,=\, \varepsilon^\alpha_{\;\beta} J^i_{\;j}\, \rho^{\beta j} \quad H^{\alpha i} \,=\, \varepsilon^\alpha_{\;\beta} J^i_{\;j} H^{\beta j}\;. \tag{3}$$

Here $\varepsilon^\alpha_{\;\beta}$ is the complex structure of Σ, verifying $\varepsilon^\alpha_{\;\beta}\varepsilon^\beta_{\;\gamma} \,=\, -\delta^\alpha_{\;\gamma}$. The greek indices are tangent indexes to Σ, they take on two values, say, 1 and 2. These indexes are raised and lowered by using a metric, $h_{\alpha\beta}$, compatible with the complex structure $\varepsilon^\alpha_{\;\beta}$. It should be mentioned that roman indices run from 1 to $\dim M$ and that they are associated to a given basis of $f^*(T)$.

The symbol Q denotes the BRST-like charge characteristic of cohomological field theories [5], which can be obtained by "twisting" [6] the appropriate N=2 supersymmetric field theory [6, 7]. The action of Q on the fields introduced above reads [1]

$$\begin{aligned}
&\{Q,u^i\} \,=\, -\chi^i, \qquad \{Q,\chi^i\} \,=\, 0\\
&\{Q,\rho^{\alpha i}\} \,=\, i\Big(H^{\alpha i} + \frac{1}{2} i\varepsilon^\alpha_{\;\beta} D_k J^i_{\;j}\chi^k\rho^{\beta j} - i\Gamma^i_{jk}\chi^j\rho^{\alpha k}\Big)\\
&\{Q,H^{\alpha i}\} \,=\, i\Big(-\frac{1}{4}(R^i_{jkl} - R^m_{nkl}J^i_{\;m}J^n_{\;j})\chi^k\chi^l\rho^{\alpha j} + \frac{i}{2}\varepsilon^\alpha_{\;\beta} D_k J^i_{\;j}\chi^k H^{\beta j} +\\
&\frac{1}{4} D_k J^i_{\;m} D_l J^m_{\;\;j}\chi^k\chi^l\rho^{\alpha j} - i\Gamma^i_{jk}\chi^j H^{\alpha k}\Big)\;.
\end{aligned} \tag{4}$$

In the preceding equations Γ^i_{jk} stands for the Levi-Civita connection on M and R^i_{jkl} denotes the Riemann tensor for this connection.

Besides the BRST-like symmetry Q, the model whose action is displayed in eq. (2) has at the classical level a $U(1)$ symmetry which obeys $[U,Q]=0$. The fields u, χ, ρ and H have, respectively, the following $U(1)$ quantum numbers: 0, 1, -1 and 0. The action $\mathcal{S}$ is conformal invariant and it has $U \,=\, 0$.

Following ref. [3] we next introduce an anticommuting variable θ with conformal spin zero and $U(1)$ charge charge -1, and, define the following superfields

$$\phi^i(\sigma,\theta) \;=\; u^i(\sigma) + i\,\theta\chi^i(\sigma)\;, \tag{5}$$

$$P^{\alpha i}(\sigma,\theta) = \rho^{\alpha i} + \theta\left(H^{\alpha i} - \frac{1}{2} i D_k J^i_{\;j} J^j_{\;l}\chi^k\rho^{\alpha l} - i\Gamma^i_{jk}\chi^j\rho^{\alpha k}\right)\,. \tag{6}$$

The superfields ϕ^i have boths $U(1)$ charge and conformal spin 0. The anticommuting superfield $P^{\alpha i}$, which is constrained by a selfduality equation analogous to eq. (3), has $U(1)$ charge -1 and conformal spin 1. The action of Q on the the superfields in eqs. (5) and (6) is given by $\frac{\partial}{\partial\theta}$.

We are now ready to establish a superspace formulation of our action. It is not difficult to show that the action in eq. (1) can be recast into the following form

$$\mathcal{S}(\kappa_1,\kappa_2) \;=\; \int d^2\sigma d\theta\,\Big(-\frac{\kappa_1}{4}\,P^{\alpha i} D_\theta P_\alpha^{\;j}\,g_{ij}(\phi) + \kappa_2\,P^{\alpha i}\partial_\alpha\phi^j\,g_{ij}(\phi)\Big)\,, \tag{7}$$

where $D_\theta P^{\alpha i} \;=\; \partial_\theta P^{\alpha i} + \partial_\theta\phi^j\Gamma^i_{jk}P^{\alpha k}$. One recovers the superspace action of ref. [3] by setting $\kappa_1 \;=\; \kappa_2 \;=\; 1$ in eq. (7), whereas $\kappa_1 \;=\; 0, \kappa_2 \;=\; 1$ corresponds to a superspace formulation of the "delta gauge" action introduced in ref. [4]. Notice that the action in eq. (7) has $U \;=\; 0$ and that it is superconformal invariant, as required.

Our next move will be the computation of the beta functions of our model for generic values of κ_1 and κ_2. We shall quantize the model by using the background field method [8]. It is a lengthy, though straightforward, computation to carry out the expansion of the action $\mathcal{S}(\kappa_1,\kappa_2)$ around the isolated background field configurations $\overline{\phi}^i$ and $\overline{P}^{\alpha i}$; the latter corresponding to the full quantum superfields ϕ^i and $P^{\alpha i}$, respectively. However, to unveil the one-loop ultraviolet divergent structure of our model, and carry out its renormalization, we need only to consider the following contributions

$$\overline{\mathcal{S}}(\kappa_1,\kappa_2) \;=\; \int d^2\sigma d\theta\,\Big(-\frac{\kappa_1}{4}\,\overline{P}^{\alpha i} D_\theta \overline{P}_\alpha^{\;j}\,g_{ij}(\overline{\phi}) + \kappa_2\,\overline{P}^{\alpha i}\partial_\alpha\overline{\phi}^j\,g_{ij}(\overline{\phi})\Big)\,, \tag{8}$$

$$\mathcal{S}_{\rm prop}(\kappa_1,\kappa_2) \;=\; \int d^2\sigma d\theta\,\Big(-\frac{\kappa_1}{4}\,\mathcal{P}^{\alpha i} D_\theta \mathcal{P}_\alpha^{j}\,g_{ij}(\overline{\phi}) + \kappa_2\,\mathcal{P}^{\alpha i} D_\alpha\xi^j\,g_{ij}(\overline{\phi})\Big)\,, \tag{9}$$

$$\begin{aligned}\mathcal{S}_{\rm int}(\kappa_1,\kappa_2) \;=\; &\int d^2\sigma d\theta\,\Big\{\Big[-\frac{\kappa_1}{4}\,\overline{P}^{\alpha i} D_\theta\overline{P}^j_\alpha\left(-\frac{1}{2}J^m{}_i D_k D_l J_{jm} + \frac{1}{3}R_{iklj}\right.\\
&\left.-\frac{1}{2}D_k J^m{}_i D_l J_{jm}\right) + \kappa_2\,\overline{P}^{\alpha i}\partial_\alpha\overline{\phi}^j\left(-\frac{1}{4}J^m{}_i D_k D_l J_{jm} + \frac{7}{12}R_{iklj}\right.\\
&\left.+\frac{1}{12}R_{mkln}J^n{}_i J^m{}_j - \frac{1}{8}D_k J^m{}_i D_l J_{jm}\right)\Big]\xi^k\xi^l +\\
&J^m{}_k D_l J^i_{\;m}\left(\frac{\kappa_1}{2}\,\overline{P}^{\alpha k} D_\theta\mathcal{P}_\alpha^{\;j} - \frac{\kappa_2}{2}\,\mathcal{P}^{\alpha k}\partial_\alpha\overline{\phi}^j\right)g_{ij}(\overline{\phi})\xi^l + \kappa_2\,D_k\overline{P}^{\alpha i} D_\alpha\xi^j\,\xi^k\,g_{ij}(\overline{\phi})\Big\}\,. \end{aligned} \tag{10}$$

The fields ξ^i and $\mathcal{P}^{\alpha i}$ embody the quantum fluctuations around the background fields $\overline{\phi}^i$ and $\overline{P}^{\alpha i}$, respectively. In the background field quantization procedure, one integrates over ξ^i and $\mathcal{P}^{\alpha i}$ to obtain the effective action [3].

Integrating out the fields ξ^i and $\mathcal{P}^{\alpha i}$ in $\mathcal{S}_{\rm int}$ with the Boltzman factor provided by $\mathcal{S}_{\rm prop}$ one easily obtains the following one-loop ultraviolet divergent contribution to the effective action $\Gamma[\overline{\phi},\overline{P}]$

$$\Gamma_{\rm div}[\overline{\phi},\overline{P};\kappa_1,\kappa_2] \;=\; \int d^2\sigma\, d\theta \left(-\frac{\kappa_1}{4}\,\overline{P}^{\alpha i} D_\theta \overline{P}^j_\alpha\, \mathcal{K}^{(1)}_{ij} + \kappa_2\, \overline{P}^{\alpha i}\partial_\alpha\overline{\phi}^j\, \mathcal{K}^{(2)}_{ij}\right), \qquad (11)$$

with

$$\begin{aligned}
\mathcal{K}^{(1)}_{ij} &= \frac{(\kappa_1/\kappa_2^2)}{2\pi\epsilon}\Big(-\frac{1}{2}J^m{}_i D_k D^k J_{jm} - \frac{1}{3}R_{ij} - \frac{1}{2}D_k J^m{}_i D^k J_{jm}\Big)\,,\\
\mathcal{K}^{(2)}_{ij} &= \frac{(\kappa_1/\kappa_2^2)}{2\pi\epsilon}\Big(-\frac{1}{4}J^m{}_i D_k D^k J_{jm} + \frac{7}{12}R_{ij} - \frac{1}{8}D_k J^m{}_i D^k J_{jm} - \frac{1}{2}R_{mn}J^n{}_i J^m{}_j\Big)\,. \qquad (12)
\end{aligned}$$

The symbol ϵ stands for the standard dimensional regularization regulator. We have taken the manifold Σ to be flat. The parameter κ_2 cannot be set to zero, otherwise the free propagator will not exist.

Eqs. (8), (11) and (12) furnish the tree-level and one-loop ultraviolet divergent contributions to the effective action. To subtract these divergences we will first express the bare objects g_{ij}, J^i_j and $\overline{P}^{\alpha i}$ in terms of the corresponding renormalized objects

$$g_{ij} \;=\; \mu^{-\epsilon}(g^{(r)}_{ij} - \delta g_{ij})\,, J^i_j \;=\; \mu^{-\epsilon}(J^{(r)\,i}{}_j - \delta J^i_j)\,, \overline{P}^{\alpha i} \;=\; \overline{P}^{(r)\,\alpha i} - \delta\overline{P}^{\alpha i}\,, \qquad (13)$$

and, then, we will substitute eqs. (13) back in eq. (8). The symbol μ of eq. (13) being the renormalization scale. As it turns out, renormalization is achieved if

$$\begin{aligned}
\delta g_{ij} &= \frac{(\kappa_1/\kappa_2^2)}{2\pi\epsilon}\left(\frac{5}{6}R_{ij} + \frac{1}{6}J^k_i R_{kl} J^l_j - \frac{1}{4}D_k J^l_i D^k J_{jl}\right),\\
\delta\overline{P}^{\alpha i} &= \frac{(\kappa_1/\kappa_2^2)}{2\pi\epsilon}\overline{P}^{\alpha k}\left(\frac{1}{4}J^m{}_k D_l D^l J^i{}_m - \frac{1}{4}R^i_k - \frac{1}{12}J^m{}_k R_{ml}J^{li} + \frac{3}{8}D_m J^l_k D^m J^i_l\right). \qquad (14)
\end{aligned}$$

The fact that both $\overline{P}^{\alpha i}$ and $\overline{P}^{(r)\,\alpha i}$ ought to be selfdual leads to the following one-loop constraint

$$\delta\overline{P}^{\alpha i} \;=\; \varepsilon^\alpha_{\;\beta} J^{(r)\,i}{}_j\,\delta\overline{P}^{\beta j} + \varepsilon^\alpha_{\;\beta}\,\delta J^i_j\,\overline{P}^{(r)\,\beta j}$$

By solving the preceding equation for δJ^i_j, one obtains

$$\delta J^i_j \;=\; \frac{(\kappa_1/\kappa_2^2)}{2\pi\epsilon}\Big(-\frac{1}{3}R^i_k J^{(r)\,k}{}_j + \frac{1}{3}J^{(r)\,i}{}_k R^k_j + \frac{1}{2}D^k D_k J^{(r)\,i}{}_j - \frac{3}{4}J^i_l D^m J^{(r)\,l}{}_k D_m J^{(r)\,k}{}_j\Big)\,. \qquad (15)$$

One may show that both the bare tensor J^i_j and the renormalized tensor $J^{(r)\,i}{}_j$ are almost complex structures over M, modulo two-loop corrections. Indeed, if, say, $J^{(r)\,i}{}_j$ is an almost

complex structure over M, the correction $\delta J^i_{\ j}$ in eq. (15) obeys the one-loop consistency condition

$$J^{(r)\,i}_{\quad\ k}\,\delta J^k_j + \delta J^i_{\ k}\,J^{(r)\,k}_{\quad\ j} \;=\; 0.$$

We next introduce the beta function β^g_{ij} of the metric and the beta function $\beta^{J\,i}_{\ \ j}$ of the almost complex structure as usual:

$$\beta^g_{ij} \;=\; \mu\,\frac{\partial g^{(r)}_{ij}}{\partial\mu},\;\beta^{J\,i}_{\ \ j} \;=\; \mu\,\frac{\partial J^{(r)\,i}_{\quad\ j}}{\partial\mu}$$

Eqs. (13), (14) and (15) lead to

$$\begin{aligned}
\beta^g_{ij} \;&=\; -\frac{(\kappa_1/\kappa_2^2)}{2\pi}\Big(\frac{5}{6}R_{ij} + \frac{1}{6}J^k_{\ i}R_{kl}J^l_{\ j} - \frac{1}{4}D_kJ^l_{\ i}D^kJ^{jl}\Big)\,,\\
\beta^{J\,i}_{\ \ j} \;&=\; -\frac{(\kappa_1/\kappa_2^2)}{2\pi}\Big(-\frac{1}{3}R^i_kJ^k_{\ j} + \frac{1}{3}J^i_{\ k}R^k_j + \frac{1}{2}D^kD_kJ^i_{\ j} - \frac{3}{4}J^i_{\ m}D^kJ^m_{\ \ l}D_kJ^l_{\ j}\Big)\,. \qquad (16)
\end{aligned}$$

We have dropped the superscript r from all objects in the preceding equations to simplify the notation; they are renormalized objects though.

Let us summarize. We have shown that both beta functions β^g_{ij} and $\beta^{J\,i}_{\ \ j}$ depend on the gauge parameters κ_1 and κ_2. These parameters are the coefficients of the two Q-exact terms that constitute the classical action our model. The beta functions are thus "gauge" dependent artifacts. Their value should not affect, therefore, the vacuum expectation values of the observables of the model. Notice that if we set $\kappa_1 = \kappa_2 = 1$ in eq. (16) we will retrieve the beta functions for the action in ref. [1], which were computed in ref. [3]. If we send k_1 to zero, so as to obtain the "delta gauge" action, the beta functions in eq. (16) will also approach zero. We have thus shown that the beta funstions in ref. [3] can be connected to the beta functions in ref. [4] by means of smooth curves. Also notice that, at variance with topological Yan-Mills theories [2], the renormalization of the model at hand cannot be carried out by a mere renormalization of its "gauge" parameters κ_1 and κ_2. We would also like to mention that the counterterm structure we have worked out is consistent with a Mathai-Quillen interpretation of the renormalized theory, provided the unregularized model have such an interpretation [9].

A final comment. Since we have computed one-loop beta functions, we have not paid any attention to a rigorous discussion of the regularization of the model by means of dimensional regularization. Higher loop computations will certainly demand such a discussion [10].

Acknowledgments

C.P. Martín acknowledges partial finacial support from CICyT.

References

[1] E. Witten, Comm. Math. Phys. **118** 411(1988).

[2] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, Phys. Rep. **209** 129(1991).

[3] J. H. Horne, Nucl. Phys. **B318** 22(1989).

[4] D. Birmingham and M. Rakowski, Modd. Phys. Lett. **A 6** 129(1991).

[5] E. Witten, Int. J. Mod. Phys. **A 6** 2775(1991).

[6] J.M.F. Labastida and P.M. Llatas, Nuc. Phys. **B379** 220(1992).

[7] N.D. Lambert, Nucl. Phys. **B445** 169(1995).

[8] L. Alvarez-Gaumé, D.Z. Freedman and S. Mukhi, Ann. Phys. **134** 85(1981); S. Mukhi, Nucl. Phys. **B264** 640(1985).

[9] S. Cordes, G. Moore and S. Ramgoolam, *Lectures on 2-d Yang-Mills theory, equivariant cohomology and topological field theories*, `hep-th/9411210`.

[10] M. Bos, Phys. Lett. **B189** 435(1987); H. Osborn, Ann. Phys. **200** 1(1990).